**Green's functions**

Again we’ll look at the 2nd order linear non-constant coefficient inhomogeneous ODE:



We saw that the transform method and the eigenfunction expansion method could both be used to solve an inhomogeneous equation in terms of a so-called Green’s function. We’ll examine this approach in a little more detail now.

In this method you find the general solution by finding a Green's function which satisfies the boundary conditions or initial conditions as appropriate. The differential operator need not be Hermitian. Similarly to above. If it's too difficult imposing the boundary conditions on the Green's function, you can replace it with whatever conditions you like - or no conditions possibly - and just use the Green's function as a particular solution. Then find the general homogeneous solution and impose, again, that y = yh + yp obey the desired boundary conditions.

**Initial Condition Green’s Functions**

This presumes we have a problem like:



If our initial conditions aren’t homogeneous, then we can either use a substitution y -> y + mx + b to make the conditions homogeneous, or we can just consider our resultant solution simply a particular solution. Now we presume a solution of the form.



And we see that: G(x,z) must satisfy the equation:



Practically, we will just find G(x<z) and G(x>z), both solutions to the homogeneous equation, and impose the initial condition on G(x<z): G(x = 0, z). Often this will require G(x<z) = 0. The Green’s Function will obey boundary conditions at x = z. Given any discontinuity and stuff, G´´ would be the most singular. So G´´ must go as a delta function, which means G´ must go as a step function and have a unit step up discontinuity as we cross x = z, and G must be continuous.

So our boundary conditions are:



If we had a first order ODE, like,



Then G´ would have to be like a delta function, and so it would have to be discontinuous across the boundary:



**Example**

For example, let’s do:



If we use the integrating factor method we get:



Now let’s do the GF approach. First we solve the homogeneous equation with initial conditions:



Then we do the particular solution:



G(t,t´) satisfies the ODE with the delta function inhomogeneity and homogeneous boundary conditions: G(t=0) = 0.



To satisfy the initial condition we need A = 0. And to satisfy the other condition we integrate around the delta function:



So we have:



So our particular solution is:



And our entire solution is:



So this agrees.

**Boundary Value Green’s Functions**

Now we’ll consider how to get boundary value problem Green’s functions. So the ODE we’re looking at is something like this:



Again if boundary conditions are inhomogenous they can be made homogenous via the aforementioned substitution. So then we assume a solution of the form:



Plugging into the ODE we get the familiar delta function ODE. And it can be seen that if the Green’s function satisfies the BC, then y will too, but only if they're homogenous. (derivatives are w/r to x BTW):



So you must solve L[G(x,z)] = 0 in region to left and right of z , which is an element of (a,b) - since z integrates from a to b. Again the two solutions G(x<z) and G(x>z) must be matched up at z. If L[y] ~ y´´ + p(x)y´ + q(x)y, then y´´ must give us the delta function, and since there is no coefficient on y´´ it must be a unit delta function. So the integral of y´´, i.e., y´ will have a unit discontinuity at z, and y itself will be continuous. So we have:



when solving G.F. ODE in x, think of z as some constant in the interval, when integrating with respect to z, think of x as some constant in the interval - i.e., when you vary one, think of the other as being fixed somewhere in the interval.